JHEP HYPER VERSION

RECEIVED: *October 15, 2001*, ACCEPTED: *December 13, 2001*
REVISED: *November 20, 2001*

Matrix description of D-branes on 3-spheres*

Jacek Pawełczyk

Institute of Theoretical Physics, Warsaw University
Hoża 69, PL-00-681 Warsaw, Poland
E-mail: Jacek.Pawelczyk@fuw.edu.pl

Harold Steinacker

Laboratoire de Physique Théorique et Hautes Energies, Université de Paris-Sud
Bâtiment 211, F-91405 Orsay, France, and
Sektion Physik der Ludwig-Maximilians-Universität München
Theresienstr. 37, D-80333 München, Germany
E-mail: Harold.Steinacker@physik.uni-muenchen.de

ABSTRACT: We discuss a matrix model for D0-branes on $S^3 \times M^7$ based on quantum group symmetries. For finite radius of S^3, it gives results beyond the reach of the ordinary matrix model. For large radius of S^3 all known static properties of branes on S^3 are reproduced.

KEYWORDS: D-branes, M(atrix) Theories.

*Work supported by Polish State Committee for Scientific Research (KBN) under contract 5 P03B 150 20 (2001-2002).

Contents

1. Introduction

Recently it has been discovered that in a special limit the dynamics of D-branes can be described by a finite matrix model [1]. Together with works on NCG in string theory [2] this has sparked new interest in strings propagating in antisymmetric tensor fields backgrounds. We are interested here in the dynamics of D0-branes in a background of the form $S^3 \times M^7$. In the approximation of infinite radius of S^3, the dynamics is given by the Myers action [1]. This action can be applied in various situations and leads to numerous interesting effects (see e.g. [3]). Unfortunately, it can not describe the finite k effects known e.g. from the DBI action [4] or the WZW description of strings in NSNS background [5, 6, 7]. Here k is the level of the appropriate $\widehat{\mathrm{su}(2)}_k$ WZW model (see below), which is also the radius of S^3 in suitable units.

In this paper we would like to go beyond this approximation. We discuss the dynamics of D0-branes on $S^3 \times M^7$ spaces with a NSNS antisymmetric tensor field background for finite radius k of S^3. Our aim is to construct a matrix model which will give results as good as the DBI action for Dp-branes, and which might be a building block of a full D0-brane theory. Our approach is inspired by [8], based on the premise that the algebra of boundary fields is most naturally understood as transforming under the quantum group $U_q(\mathrm{su}(2))$ for $q = e^{\frac{i\pi}{k+2}}$. It is then natural to expect that the full isometry group of S^3 should be q-deformed, too.

We will construct actions by imposing such a symmetry, which is a version of the quantum group (more precisely quantum universal enveloping algebra) $U_q(\mathrm{so}(4))$. The name q-matrix model originates from this algebra. Although we do not propose any definite form of the action, the $U_q(\mathrm{so}(4))$ symmetry is restrictive enough so that we can discuss several physical properties of the model. For example, we can given an expression for the energy of the wrapping which is given by the quantum dimension of a representation, and we can solve equations of motion and obtain quantum 2-spheres.

The outline of the paper is the following. In the rest of this introduction we shall recall some facts concerning the macroscopic description of branes in terms of the DBI action and the microscopic theory of many D0-branes of Myers [1]. In the latter we emphasize the approach of [6, 7] which uses an SU(2) WZW model to derive the D0-branes action. In section 2 we shall present the $U_q(\mathrm{so}(4))$ algebra, as well as some modules and invariants. It will play the crucial role in constructing the matrix model. Then we shall define the model and discuss its predictions. Section 3 is devoted conclusions and the discussion of problems and limitations of the model. Appendices contain our conventions, some results on representations of $U_q(\mathrm{su}(2))$ and two proofs.

DBI branes in NSNS background. The string theory background of interest here has the form $S^3 \times M^7$ with some nonzero H^{NSNS} and constant dilaton. The most studied example is obtained as the near horizon limit of the F1, NS5 system (see e.g. [9]). Below we write only the relevant fields living on S^3

$$\frac{ds^2}{\alpha'} = k\, d\Omega_3^2\,, \qquad \frac{H^{NSNS}}{\alpha'} = 2k\, \epsilon_3\,, \qquad e^{2\phi} = \text{const.} \tag{1.1}$$

where k denotes the number of NS5-branes and equals to the level of the appropriate $\widehat{\mathrm{su}(2)}_k$ WZW model, and ϵ_3 is the volume element of the unit 3-sphere.

The action of the D-branes is given by the DBI expression which properly describes the dynamics of branes for large (but finite) k. In [4] classical configurations of Dp-branes embedded in (1.1) were considered. The branes were embedded in such a way that two of the dimensions of the brane wrapped $S^2 \subset S^3$ and the rest extended in M^7 forming an effective D$(p-2)$-brane. It appears that the position ϑ of the spherical brane on S^3 is quantized

$$\vartheta = \frac{\pi n}{k}\,, \qquad 0 \le n \le k\,, \tag{1.2}$$

where θ is the standard polar angle. The effective D$(p-2)$-brane has a tension which takes into account the energy of the wrapping. The expression for this tension is

$$T^{eff}_{(p-2)} = T_p\, 4\pi\alpha'\, k \sin\frac{n\pi}{k}\,. \tag{1.3}$$

We should add that the same results were also obtained for the RR background in [10].

Matrix model and $\widehat{\mathrm{su}(2)}_k$ WZW model. The above system can also be studied in the $1/k$ expansion as a matrix model of many D0-branes [1]. The leading terms of the appropriate action is obtained approximating the S^3 by a 3-plane and taking $B = \frac{1}{3}k\alpha'\epsilon_{ijk}X^i dX^j \wedge dX^k$. Then

$$S \sim T_0 \mathrm{tr}\left[1 - \frac{(2\pi^2)}{k^2}\left(\frac{1}{4}\,[X^i, X^j]^2 + \frac{i}{3}\epsilon_{ijl}X^i[X^j, X^l]\right) + \cdots\right]. \tag{1.4}$$

The equations of motion for X are solved by

$$[X^i, X^j] = -i\epsilon^{ijk} X^k \,, \tag{1.5}$$

where now X's are $N \times N$ matrices of positions of N D0-branes. For the fixed central element of the above algebra (the second Casimir) this defines the so-called fuzzy sphere [11]. It appears that the fuzzy spheres have the dynamics and quantum numbers as ordinary D2-branes described in the previous subsection [7, 4]. One can rederive (1.5) using CFT language [7]. We have decided to recall some facts from this approach as it directly leads to the q-deformed case.

The analysis of the $\widehat{\mathrm{su}(2)}_k$ WZW model with the simplest boundary conditions[1] defining branes shows that D-branes are classified by an integer $\tilde{n}$ which satisfies $0 \le \tilde{n} \le k$. The Hilbert space of the boundary CFT decomposes into irreducible representations of the affine Lie algebra $\widehat{\mathrm{su}(2)}_k$. The ground ring of the associated boundary operators $Y^I_a(x)$, (I ranges from 0 to $\min(\tilde{n}, k-\tilde{n})$, and $-I \le a \le I$) form irreducible spin I representations of the horizontal su(2) algebra. Their OPE's are given by [6]

$$Y^I_a(x) Y^J_b(y) = \sum_{K,c} (x-y)^{h_K - h_I - h_J} \begin{bmatrix} I & J & K \\ a & b & c \end{bmatrix} \begin{Bmatrix} I & J & K \\ \tilde{n}/2 & \tilde{n}/2 & \tilde{n}/2 \end{Bmatrix}_q Y^K_c(y) + \cdots \,, \tag{1.6}$$

where $q = e^{\frac{\pi i}{k+2}}$, the sum goes from $K = 0$ to $\min(I+J, k-I-J, \tilde{n}, k-\tilde{n})$, $-K \le c \le K$ $h_I = I(I+1)/(k+2)$ is the conformal weight of $Y^I_a(x)$, the first bracket denotes the Clebsch-Gordon coefficients of su(2) and the curly brackets denote the q-deformed $6J$-symbols of $U_q(\mathrm{su}(2))$.

$k \to \infty$ limit. It is clear that in $k \to \infty$ limit, $h_I \to 0$ and the q-deformed $6J$-symbols tend to the ordinary $6J$-symbols so that the OPE becomes the associative matrix algebra $Mat(\tilde{n}+1) = (\underline{1} \oplus \underline{3} \oplus \cdots \oplus (2\tilde{n}+1))$. From the point of view of strings moving in the other 7 dimensions $Mat(\tilde{n}+1)$ is a kind of Chan-Patton algebra. It carries information about the internal geometry of the brane. In fact, Y^I_a can be thought of as spherical harmonics with an unusual multiplication law. It is known as the algebra of functions of the fuzzy sphere [11]. The first descendants (gauge fields) $j^i_{-1} Y^I_a$ of the ground ring have conformal dimensions $h^{(1)}_I = 1$ in this limit. Thus they form vertex operators describing the dynamics of the wrapped D-branes. These were used in [7] in order to find the effective action (1.4) for the X^i matrices.

Finite k. Now Y^I_a do not form any algebra, as h_I are no longer zero.[2] If one naively drops factors $(x-y)^{h_I+h_J-h_S}$, then Y^I_a form a quasiassociative algebra, which is not a very convenient structure to work with.

It was noticed in [6, 8] that quantum groups come to rescue: one can recover the associativity of the algebra at the price of twisting it. Technically this means that one

[1]The boundary conditions break the current algebra $\mathrm{su}_L(2) \times \mathrm{su}_R(2) \to \mathrm{su}_V(2)$. This should be related to the form of the twisting of $U_q(\mathrm{so}(4))$ we have chosen in section 2.1 but the precise connection is not clear to us.

[2]Some of the operators can have large dimensions: for maximal $I \sim k/2$ this can be about $h \sim k/4$.

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changes the ordinary Clebsch-Gordon coefficients in (1.5) by their q-deformed versions. Then Y_a^I form representations of $U_q(\mathrm{su}(2))$ (which will be referred to as $U_q(\mathrm{su}(2))^V$ below). For $\tilde{n} \in (0, [k/4])$, the tensor product of any two elements Y_a^I is completely reducible, and after this twisting one obtains the associative algebra $Mat(\tilde{n}+1) = (\underline{1} \oplus \underline{3} \oplus \cdots \oplus (2\tilde{n}+1))$. For $\tilde{n} > [k/4]$ this is no longer the case;[3] however our results below continue to make sense even for $\tilde{n} > [k/4]$, and will be manifestly invariant under $\tilde{n} \to k - \tilde{n}$. The position matrices (analogs of X^i) correspond again to the gauge field operators $j_{-1}^i Y_a^I$. They are also vectors in $\underline{3}$ of $U_q(\mathrm{su}(2))^V$.

Below we shall significantly extend this approach and analyze its physical content.

2. q-matrix model

This section is devoted to the construction and the analysis of a matrix model based on a certain q-deformed symmetry algebra. Our basic variables are position matrices M^μ ($\mu = 1, \ldots, 4$) describing a "quantum 3-sphere" S_q^3 and representing positions of a system of $N = \tilde{n}+1$ objects which we shall call q-D0-branes.[4] The physical interpretation of these fields is similar to X^i's of (1.4). Under the "internal" $U_q(\mathrm{su}(2))^V$ they shall transform in the quantum adjoint action

$$U_q(\mathrm{su}(2))^V \ni u \triangleright M^\mu = u_1 M^\mu S(u_2)\,, \tag{2.1}$$

where $u_{1,2}$ are $\tilde{n}+1$ dimensional matrices obtained by taking the representation of the coproduct $\Delta(u) = u_1 \otimes u_2$, and S is the antipode (see appendix A). The position matrices M^μ belong also to the $\underline{4}$-dimensional module of the space-time symmetry algebra which will be a twisted version of $U_q(\mathrm{so}(4))$ denoted by $U_q(\mathrm{so}(4))_{\mathcal{F}}$: this will be an isometry of the "quantum 3-sphere" S_q^3. We shall also assume that classical single brane configuration will break $U_q(\mathrm{so}(4))_{\mathcal{F}}$ to $U_q(\mathrm{su}(2))^V$ of the previous section (see footnote 1). Technically this means that $U_q(\mathrm{su}(2))^V$ should be Hopf subalgebra of $U_q(\mathrm{so}(4))_{\mathcal{F}}$. This will be crucial for the predictive power of the model. From now on, all group-theoretic objects (invariant tensors etc.) are understood to carry a label q, which is suppressed. For the notations we refer to appendix B and [8].

Before we go into the details we should make a comment. One must be aware that $q = e^{\frac{\pi i}{k+2}}$ involves k, which is inverse proportional to the curvature of the S^3. This means that the process of changing symmetry algebras to their q-deformed versions takes into account the so-called sigma model corrections of the string theory, which are small for large k. On the other hand, our analysis is based only on the S^3 part of the model, and it seems that one can not hope to get the full answers for finite k. Hence the situation is unclear at the moment, and the results obtained below should be treated with some caution. In the end of this section we shall compare the q-matrix model results with some results obtained by other means for large k and show their astonishing agreement.

[3]For $\tilde{n} > [k/4]$, quasiassociativity cannot be removed completely due to the truncation in the above algebra. We will not worry here about the precise construction of fields in general.

[4]We add the prefix "q-" here because it might be that these objects are different than the usual D0-branes.

2.1 $U_q(\mathrm{so}(4))$

Algebra. The aim of this section is to construct a suitable quantum "group" (Hopf algebra) which describes the symmetries of a quantum 3 sphere S^3_q, formally defined by (2.11), and which is compatible with a "vector" $U_q(\mathrm{su}(2))^V$ symmetry. We expect that this isometry algebra should be a version of the quantum group $U_q(\mathrm{so}(4))$. As an algebra, the "standard" $U_q(\mathrm{so}(4))$ is simply the tensor product of two commuting $U_q(\mathrm{su}(2))$ algebras, $U_q(\mathrm{so}(4)) = U_q(\mathrm{su}(2))^L \otimes U_q(\mathrm{su}(2))^R$. It carries naturally the structure of a (quasitriangular) Hopf algebra. There is a natural embedding of the "vector" (sub)algebra $U_q(\mathrm{su}(2))^V \to U_q^L \otimes U_q^R$:

$$d : u \mapsto (u_1 \otimes u_2) = \Delta(u)\,, \tag{2.2}$$

where $\Delta(u)$ is the coproduct of $U_q(\mathrm{su}(2))$. The problem with this definition is that the embedding d is not compatible with the standard coproduct of $U_q(\mathrm{su}(2))^L \otimes U_q(\mathrm{su}(2))^R$. This means that $U_q(\mathrm{su}(2))^V$ will not act correctly on products of fields (i.e. matrices, here).

The solution to this problem is provided by twisting. For the general theory of twisting we refer to [13], and to [14] (section 2.3). Consider the modified coproduct

$$\begin{aligned} \Delta_{\mathcal{F}} : U_q^L \otimes U_q^R &\to (U_q^L \otimes U_q^R) \otimes (U_q^L \otimes U_q^R), \\ u^L \otimes u^R &\mapsto \mathcal{F}(u_1^L \otimes u_1^R) \otimes (u_2^L \otimes u_2^R)\mathcal{F}^{-1}\,, \end{aligned} \tag{2.3}$$

where $\mathcal{F} = \mathcal{R}_{32} \in U_q \otimes U_q$ is the universal R-"matrix" of $U_q(\mathrm{su}(2))$ with reversed components. The subscripts of $\mathcal{R}$ refer to the positions in the tensor product. This defines again a quantum group, i.e. a quasitriangular Hopf algebra. We will denote this (twisted) quantum group with $U_q(\mathrm{so}(4))_{\mathcal{F}}$. It is easy to check that it satisfies

$$\Delta_{\mathcal{F}} \circ d = (d \otimes d) \circ \Delta\,, \tag{2.4}$$

i.e. $U_q(\mathrm{so}(4))_{\mathcal{F}}$ is compatible with the embedding (2.2) of the quantum group $U_q(\mathrm{su}(2))^V$. This is what we were looking for. In fact if q were real, this $U_q(\mathrm{so}(4))_{\mathcal{F}}$ would coincide precisely with the q-Lorentz group $U_q(\mathrm{so}(3,1))$ [16], and much is known about its algebraic structure and representations. However since we are looking for matrix models in a rather different setting, we shall construct the modules of interest (matrices) and the corresponding invariants using an ad-hoc approach, trying to minimize the technical background.

Modules. Here we discuss the $\underline{4}$ module of $U_q(\mathrm{so}(4))_{\mathcal{F}}$ to which the position matrices M^μ belong. There are two obvious ways to construct it: consider 2×2 matrices $X = X^{\dot\alpha}{}_{\beta}$ and $\tilde{X} = \tilde{X}^{\alpha}{}_{\dot\beta}$, where dotted indices refer to U_q^L and undotted indices to U_q^R. The naive guess would be that they transform as $X \to u^L X S(u^R)$ and $\tilde{X} \to u^R \tilde{X} S(u^L)$. However, this is not compatible with the requirement that products of matrices transform using the coproduct of $U_q(\mathrm{so}(4))_{\mathcal{F}}$, i.e. that these matrices form a $U_q(\mathrm{so}(4))_{\mathcal{F}}$-module algebra. However, the following action of $U_q(\mathrm{so}(4))_{\mathcal{F}}$ defines a consistent $U_q(\mathrm{so}(4))_{\mathcal{F}}$-module algebra:

$$(X, \tilde{X}) \mapsto u \triangleright_{\mathcal{F}} (X, \tilde{X}) = \Big(u^L X S(u^R), \mathcal{R}_2 u^R \mathcal{R}_b^{-1} \tilde{X} S(\mathcal{R}_1 u^L \mathcal{R}_a^{-1})\Big) \tag{2.5}$$

extended to products via the new coproduct (2.3). One verifies that products transform as

$$X\tilde{X} \to u_1^L X\tilde{X} S(u_2^L)\,, \qquad \tilde{X}X \to u_1^R \tilde{X} X S(u_2^R)\,, \tag{2.6}$$

see appendix C. The vector subalgebra $U_q(\mathrm{su}(2))^V$ then acts as

$$u^V \triangleright_{\mathcal{F}} (X, \tilde{X}) = \left(u_1^V X S(u_2^V), u_1^V \tilde{X} S(u_2^V)\right), \tag{2.7}$$

as it should. Also, $X\tilde{X} \to u_1^V X\tilde{X} S(u_2^V)$ and $\tilde{X}X \to u_1^V \tilde{X}X S(u_2^V)$. This shows that all the objects (algebras and modules) have the required properties.

Since there is only one representation $\underline{4}$ of $U_q(\mathrm{so}(4))_{\mathcal{F}}$, there should be a relation between these two actions on X and $\tilde{X}$. Indeed, one can identify X and $\tilde{X}$ via

$$\tilde{X} = -\mathcal{R}_2 q^H S(X) S(\mathcal{R}_1). \tag{2.8}$$

Here S is the induced antipode of $U_q(\mathrm{su}(2))$ on the 2×2 matrix X. It is easy to verify that this indeed induces the action (2.5) on $\tilde{X}$. This restricts the degrees of freedom as required, and will allow us to define a suitable quantum 3-sphere. We use this to construct a basis for $\tilde{X}$ modules in terms of the basis of X modules. Let us call the latter $\sigma_\mu = (iq, \sigma_i)$. Then the former called $\tilde{\sigma}_\mu$ are given by $\tilde{\sigma}_\mu = (-iq^{-1/2}, q^{1/2}\sigma_i)$ (see appendix C).

Invariants. Using these results, one sees immediately that q-traces ($\mathrm{tr}_q(A) \equiv \mathrm{tr}(Aq^{-H})$)

$$S_1^{(n)} = \mathrm{tr}_q((X\tilde{X})^n) \quad S_2^{(n)} = \mathrm{tr}_q((\tilde{X}X)^n), \qquad n \in \mathbb{N} \tag{2.9}$$

are invariant under $U_q(\mathrm{so}(4))_{\mathcal{F}}$. Explicitly, in case of interest (2×2 matrices X, $\tilde{X}$) we have $q^{-H} = \mathrm{diag}(q, q^{-1})$. For example, the invariant tensor for the representation $\underline{4}$ is obtained by $q^{1/2}[2]\, g_{\mu\nu} = \mathrm{tr}_q(\sigma_\mu \tilde{\sigma}_\nu)$. It has the block diagonal form $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & g_{ij} \end{pmatrix}$ where g_{ij} is the $U_q(\mathrm{su}(2))$ invariant metric defined as above from σ_i matrices and given explicitly in appendix B.

2.2 The model and its comparison with known results

We would like to construct a "quantum" 3-sphere S_q^3 which will play the role of the S^3 from the string geometry (1.1). Thus we consider matrix valued $Mat(\tilde{n}+1)$ fields M^μ in $\underline{4}$ of $U_q(\mathrm{so}(4))_{\mathcal{F}}$ living on a S_q^3. Using those variables we shall construct candidates for actions describing the dynamics of (quantum) D0-branes.

The techniques of the previous subsection give a convenient way of creating $U_q(\mathrm{so}(4))_{\mathcal{F}}$-invariant tensors, with the help of (2.9). Let us introduce

$$M = \sigma_\mu M^\mu, \qquad \widetilde{M} = \tilde{\sigma}_\mu M^\mu. \tag{2.10}$$

Then the formal "quantum" 3-sphere S_q^3 is defined by the relation $\mathrm{tr}_q(M\widetilde{M}) = q^{1/2}[2]R^2$, i.e.

$$(M^4)^2 + g_{ij} M^j M^j = R^2 \cdot 1, \qquad R > 0. \tag{2.11}$$

It is natural to expect that the radius of this[5] sphere is $R^2 \sim k$ in analogy with the string background (1.1).

[5]We shall not discuss its precise algebraic/geometric meaning.

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We assume that the action of the q-matrix model can be expressed as power series in M and $\widetilde{M}$ (see (2.9)). Due to the above constraint we have

$$M\widetilde{M} = q^{1/2}[R^2 + F_L^l \sigma_l]\,, \qquad \widetilde{M}M = q^{1/2}[R^2 + F_R^l \sigma_l]\,, \tag{2.12}$$

where

$$\begin{aligned} F_L^l &= i(qM^4M^l - q^{-1}M^lM^4) - \epsilon_{ij}^l M^iM^j \\ F_R^l &= i(-q^{-1}M^4M^l + qM^lM^4) - \epsilon_{ij}^l M^iM^j \end{aligned} \tag{2.13}$$

are vectors of $U_q(\mathrm{su}(2))^L$ and $U_q(\mathrm{su}(2))^R$, respectively. We call them L (R) field-strength as, in the limit $q = 1$, they are equal and correspond to the ordinary field strength of [7]. But notice that we can not write any Chern-Simons term here. From (2.12) it follows that any polynomial in M and $\widetilde{M}$ can be also written in terms of $F_{L,R}^l$. The examples are $g_{ij}F_L^iF_L^j$, $g_{ij}F_R^iF_R^j$, $\epsilon_{ij}^n F_{Ln}F_L^iF_L^j$, where ϵ is an invariant 3-index tensors of $U_q(\mathrm{su}(2))^{L,R} \subset U_q(\mathrm{so}(4))_{\mathcal{F}}$ (see appendix B).

Now we can easily construct the invariant actions. One must remember that $F_{L,R}^l \in Mat(\tilde{n}+1)$ i.e. they transforms under $U_q(\mathrm{su}(2))^V$. Explicitly[6]

$$U_q(\mathrm{su}(2))^V \ni u \triangleright F_{L,R}^l = u_1 F_{L,R}^l S(u_2)\,. \tag{2.14}$$

The invariants are expressed as q-traces $\mathrm{tr}(Aq^{-H})$. Now the trace is over the "internal" degrees of freedom, i.e. over $Mat(\tilde{n}+1)$. The matrix H is determined by the representation we want to work with. This is a somewhat unusual and unpleasant property (compared to the standard matrix model) of the model, but at the moment we do not know how to overcome it. The representations can be reducible (describing sets of clusters of q-D0-branes) or irreducible. In the latter case it has the following form $H = \mathrm{diag}(\tilde{n}, \tilde{n} - 2, \ldots, -\tilde{n})$.

The simplest terms which could contribute to the action are of the from

$$\mathrm{tr}_q(1)\,, \qquad \mathrm{tr}_q(g_{ij}F_L^iF_L^j)\,, \qquad \mathrm{tr}_q(g_{ij}F_R^iF_R^j), \qquad \mathrm{tr}_q(\epsilon_{ij}^n F_{Ln}F_L^iF_L^j), \;\ldots\,. \tag{2.15}$$

We shall not try to find out the precise form of the relevant action. This would require a detailed analysis of the string theory on $S^3 \times M^7$ which, to our knowledge, is not available now. Thus we shall limit the discussion to the most basic properties of *any action* which can be formed in our case.

First of all we notice that

$$F_L^l = F_R^l = 0 \tag{2.16}$$

solves any possible equations of motion. Moreover (2.16) nullifies all of the terms (2.15) except $\mathrm{tr}_q(1)$. The energy of the classical configuration which corresponds to the irreducible representation is therefore given by so-called quantum dimension

$$\mathrm{tr}_q(1) = [\tilde{n}+1]_q = \frac{\sin(\frac{(\tilde{n}+1)\pi}{k+2})}{\sin(\frac{\pi}{k+2})}\,. \tag{2.17}$$

[6]This can be reconciled with the concept of a differential calculus using the concept of frames, see [8].

For $1 \ll \tilde{n} \ll k$ this gives (up to an overall constant) the same function of k as appearing in the tension of the effective D$(p-2)$-brane (1.3). One should notice that the formula (2.17) is well behaved for all $\tilde{n} \in (0,k)$. Also for $\tilde{n}=0$ and $\tilde{n}=k$ it gives 1 thus it should be interpreted as the ratio of tensions of effective (D-2)-branes formed by a bound state of $N=\tilde{n}+1$ D0-branes and the tension of a single D0-brane. It is interesting to note that the same result can be obtained by CFT methods (see the first paper in [4]).

The equations of motion (2.16) can be rewritten in more explicit form[7]

$$\begin{aligned} [M^4, M^l] &= 0\,, \\ \epsilon^l_{ij}\; M^i M^j &= i(q-q^{-1}) M^4 M^l\,. \end{aligned} \tag{2.18}$$

Due to $[M^4, M^l]=0$ we can diagonalize M^4 (hence also $g_{ij}M^iM^j$). The algebra generated by solutions of (2.18) of the form $\{M^4 \sim \mathbf{1}, M^i\}$ is called the q-fuzzy sphere $S^2_{q,\tilde{n}}$ [8]. This is a slight generalization of a Podles sphere [15], q being a phase. It is characterized by the radius $r_{\tilde{n}}$ of the sphere defined by $r^2_{\tilde{n}} = g_{ij}M^iM^j$. One finds

$$r^2_{\tilde{n}} = R^2\left(1 - \frac{[2]^2_{q^{\tilde{n}+1}}}{[2]^2_q}\right) = R^2\left(1 - \frac{\cos^2(\frac{(\tilde{n}+1)\pi}{k+2})}{\cos^2(\frac{\pi}{k+2})}\right). \tag{2.19}$$

Notice that this formula is $\tilde{n} \to k-\tilde{n}$ invariant. The corresponding 4-component is naturally given by $M^4 = R\,[2]_{q^{\tilde{n}+1}}/[2]_q = R\cos(\frac{(\tilde{n}+1)\pi}{k+2})/\cos(\frac{\pi}{k+2})$, which covers positive and negative values. For large k and $1 \ll \tilde{n} \ll k$, (2.19) is well approximated by $r^2_{\tilde{n}} \approx R^2\sin^2(\frac{\tilde{n}\pi}{k})$ which is the classical expression for the radius of $S^2 \subset S^3$ as follows from (1.2) for $R^2=k$. Again we get astonishing agreement although the formula (2.19) has completely different origin, namely the representation theory of quantum algebras.

Let us also make the approximation of small spheres $S^2_{q,\tilde{n}}$ and substitute $M^4 = \sqrt{k} + O(g_{ij}M^iM^j)$ (we set $R^2=k$ here). Then $x^l = iM^l/((q-q^{-1})\sqrt{k})$ respects q-fuzzy sphere equation as written in [8],

$$\epsilon^l_{ij}\; x^i x^j = x^l\,. \tag{2.20}$$

In the large k approximation the above becomes the ordinary fuzzy sphere (1.5).

We have also checked the spectra of the fluctuations and they slightly differ from what is known for the WZW model [4] and the DBI action [10]. However the comparison we have just made might be too naive, because in order to find masses we need also the time component of the dynamics which is beyond the scope of this work.

3. Problems and conclusions

We conclude this paper with some comments concerning the obtained results. It seems astonishing that such a simple model based on almost no physical input precisely reproduces *static* properties of the D0-brane system. The origin of this success lies in the applied symmetries. One can see that formula (2.17) follows from the assumption that $Mat(\tilde{n}+1)$ is in a representation of $U_q(\mathrm{su}(2))^V$, but it is also intimately related to the fact that $U_q(\mathrm{so}(4))_{\mathcal{F}}$ constrains the simplest equations of motion $F^i_{L,R}=0$ in such a way that the

[7] algebraically, these are the same as the relations of a "light-cone" in quantum Minkowski space [16].

other contributions to the brane energy vanishes. Next, the nice equations of motion (2.18) directly follow from the twist $\mathcal{F}$ in $U_q(\mathrm{so}(4))_{\mathcal{F}}$: the latter has been imposed on physical grounds, i.e. by requiring the unbroken $U_q(\mathrm{su}(2))^V$ symmetry. The equations of motion yield an attractive formula for the radius of $S^2_{q,\tilde{n}}$. *Last but not the least*, we must stress that all obtained formulae work for all boundary conditions $\tilde{n} \in [0, k]$ suggesting that $S^2_{q,\tilde{n}}$ are well defined even for these $\tilde{n}$ which exceeds the limit of "nice"(irreducible highest weight) representations of $U_q(\mathrm{su}(2))$.

One should expect that the model has some limits of applicability. It works for large k, but the symmetry alone is not able to pick any specific action. It seems inevitable that in order to achieve more accurate results one must understand also the dynamics of M^7 degrees of freedom. Besides one must realize that the model does not have the original isometry of S^3. We do not believe that the isometry is broken by the quantum effects (due to $q = e^{i\pi/(k+2)}$) discussed in the previous section. Hence in the final version of the model, there should be the possibility to twist back the symmetry algebra from $U_q(\mathrm{so}(4))_{\mathcal{F}}$ to $U(\mathrm{so}(4))$. Furthermore, the model requires to fix the representation of $U_q(\mathrm{su}(2))^V \subset U_q(\mathrm{so}(4))_{\mathcal{F}}$ in order to define q-traces. This sounds bizarre and awaits further clarification. Next, the model has problems with reality conditions for the M matrices. More precisely, we were not able to find any reality conditions which would be compatible with the algebra actions. Finally the gauge symmetry $\mathrm{U}(\tilde{n}+1)$ of the standard matrix models is obscured here.

In spite of all those unsettled issues the model reproduces all the static properties of the D0-brane system, with astonishing accuracy. These results are beyond reach of the standard matrix model. The key to the success lies in the quantum symmetry. Its relevance to string theory clearly deserves further studies.

Acknowledgments

The authors thank J. Lukierski and the organizers of the XXXVII Winter School of Theoretical Physics in Karpacz for hospitality and stimulating atmosphere during the school. We would like to thank Chong-Sun Chu, J. Kowalski-Glikman, H. Grosse, B. Jurco, J. Madore, and W. Zakrzewski for inspiring discussions. We are also grateful for hospitality in the Laboratoire de Physique Théorique in Orsay and Sektion Physik der Ludwig-Maximilians-Universität in München. H.S. thanks the DFG for a fellowship.

A. Basic properties of $U_q(\mathrm{su}(2))$

The basic relations of the Hopf algebra $U_q(\mathrm{su}(2))$ are

$$[H, X^{\pm}] = \pm 2X^{\pm}\,, \qquad [X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}} = [H]_q\,, \tag{A.1}$$

where the q-numbers are defined as $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The action of $U_q(\mathrm{su}(2))$ on a tensor product of representations is encoded in the coproduct

$$\begin{aligned}\Delta(H) &= H \otimes 1 + 1 \otimes H \\ \Delta(X^{\pm}) &= X^{\pm} \otimes q^{-H/2} + q^{H/2} \otimes X^{\pm}\,.\end{aligned} \tag{A.2}$$

We will use the Sweedler-notation $\Delta(u) = u_1 \otimes u_2$, where a summation convention is understood. The antipode and the counit are given by

$$\begin{aligned} S(H) &= -H\,, \qquad S(X^+) = -q^{-1}X^+\,, \qquad S(X^-) = -qX^-\,, \\ \epsilon(H) &= \epsilon(X^\pm) = 0\,. \end{aligned} \tag{A.3}$$

Moreover, the following element

$$v = (S\mathcal{R}_2)\mathcal{R}_1 q^H \tag{A.4}$$

is in the center of $U_q(\mathrm{su}(2))$, and has the eigenvalues $q^{2j(j+1)}$ on the spin j representation.

B. Representations and invariant tensors

The q-deformed sigma-matrices, i.e. the Clebsch-Gordon coefficients for $(3) \subset (2) \otimes (2)$, are given by

$$\sigma_1^{++} = \sqrt{[2]_q} = \sigma_{-1}^{--}\,, \qquad \sigma_0^{+-} = q^{\frac{1}{2}}\,, \qquad \sigma_0^{-+} = q^{-\frac{1}{2}} \tag{B.1}$$

in a weight basis. Then $(\sigma_i)^\alpha{}_\beta$ are

$$\sigma_{-1} = \begin{pmatrix} 0 & q^{\frac{1}{2}}\sqrt{[2]_q} \\ 0 & 0 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 0 \\ -q^{-\frac{1}{2}}\sqrt{[2]_q} & 0 \end{pmatrix}. \tag{B.2}$$

They satisfy

$$\sigma_i\sigma_j = -\epsilon_{ij}^k\,\sigma_k + g_{ij} \tag{B.3}$$

$$\pi(u_1)\sigma_i\pi(Su_2) = \sigma_j\pi_i^j(u)\,, \tag{B.4}$$

for $u \in U_q(\mathrm{su}(2))$, where ϵ_{ij}^k is defined below, and π denotes the appropriate representation.

The invariant tensor g^{ij} for the spin 1 representation satisfies by definition

$$\pi_k^i(u_1)\,\pi_l^j(u_2)\,g^{kl} = \epsilon(u)g^{ij} \tag{B.5}$$

for $u \in U_q(\mathrm{su}(2))$. It is given by

$$g^{1-1} = -q^{-1}\,, \qquad g^{00} = 1\,, \qquad g^{-11} = -q\,, \tag{B.6}$$

all other components are zero. Then $g_{ij} = g^{ij}$ satisfies $g^{ij}g_{jk} = \delta_k^i$, and $g^{ij}g_{ij} = q^2 + 1 + q^{-2} = [3]_q$.

The Clebsch-Gordon coefficients for $(3) \subset (3) \otimes (3)$, i.e. the q-deformed structure constants, are given by

$$\begin{aligned} \epsilon_1^{10} &= q^{-1}\,, & \epsilon_1^{01} &= -q\,, \\ \epsilon_0^{00} &= -(q - q^{-1})\,, & \epsilon_0^{1-1} &= 1 = -\epsilon_0^{-11}\,, \\ \epsilon_{-1}^{0-1} &= q^{-1}\,, & \epsilon_{-1}^{-10} &= -q\,, \end{aligned} \tag{B.7}$$

and $\epsilon_{ij}^k := \epsilon_k^{ij}$. They have been normalized such that $\sum_{ij}\epsilon_{ij}^n\epsilon_m^{ij} = [2]_{q^2}\delta_m^n$. The q-deformed totally (q-)antisymmetric tensor is defined as follows:

$$\epsilon^{ijk} = g^{in}\epsilon_n^{jk} = \epsilon_n^{ij}g^{nk}\,. \tag{B.8}$$

It is invariant under the action of $U_q(\mathrm{su}(2))$,

$$\pi_l^i(u_1)\,\pi_m^j(u_2)\,\pi_n^k(u_3)\,\epsilon^{lmn} = \epsilon(u)\epsilon^{ijk}\,. \tag{B.9}$$

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C. Some proofs

The form of $\tilde{\sigma}_\mu$ Here we derive the form of $\tilde{\sigma}_\mu$. Instead of a brute force calculation, we can argue as follows: let

$$\tilde{\sigma}_i := -\mathcal{R}_2 q^H S(\sigma_i) S(\mathcal{R}_1)\,, \tag{C.1}$$

and observe that

$$\begin{aligned} u_1 \tilde{\sigma}_i S u_2 &= -\mathcal{R}_2 u_2 q^H S(\sigma_i) S(u_1) S(\mathcal{R}_1) \\ &= -\mathcal{R}_2 q^H S(u_1 \sigma_i S u_2) S(\mathcal{R}_1) = \tilde{\sigma}_j \pi^j_i(u)\,. \end{aligned} \tag{C.2}$$

This implies that $\tilde{\sigma}_i = c\sigma_i$ for a constant $c \in \mathbb{C}$. The latter can be found e.g. by first showing

$$\begin{aligned} \tilde{\sigma}_i &= S(\mathcal{R}_1 \sigma_i q^{-H} S^{-1}(\mathcal{R}_2)) \\ &= S(\mathcal{R}_1 \triangleright \sigma_i\ \mathcal{R}_a q^{-H} S^{-1}(\mathcal{R}_2 \mathcal{R}_b)) \\ &= S(\mathcal{R}_1 \triangleright \sigma_i\ (\mathcal{R}_a S\mathcal{R}_b q^{-H}) S^{-1}\mathcal{R}_2) \\ &= S(\mathcal{R}_1^{-1} \triangleright \sigma_i\ \mathcal{R}_2^{-1})(S\mathcal{R}_b \mathcal{R}_a q^H) \\ &= S(\sigma_j L^{-j}_{\ i})v\,, \end{aligned} \tag{C.3}$$

where v is the Drinfeld-Casimir (A.4), and $L^{-j}_{\ i} = \pi^j_i(\mathcal{R}_1^{-1})\mathcal{R}_2^{-1}$. On the spin 1/2 representation, v takes the value $v = q^{-3/2}$. Using $\hat{R}^{ij}_{kl} g^{kl} = q^{-4} g^{ij}$ and $\Delta L^{-j}_{\ i} = L^{-j}_{\ k} \otimes L^{-k}_{\ i}$, one can now verify $g^{ij}\tilde{\sigma}_i\tilde{\sigma}_j = q^4 v^2 g^{ij}\sigma_i\sigma_j$, which implies $c = q^2 v = q^{1/2}$.

Proof of (2.6)

$$\begin{aligned} &(u^L \otimes u^R) \triangleright X\tilde{X} = \cdot((u_1^L \otimes R_y u_a^R R_j^{-1}) \otimes (R_x u_2^L R_i^{-1} \otimes u_b^R)) \triangleright_{\mathcal{F}} (X \otimes \tilde{X}) \\ &= u_1^L X S(R_y u_a^R R_j^{-1}) R_\beta u_b^R R_\delta^{-1} \tilde{X} S(R_\alpha R_x u_2^L R_i^{-1} R_\gamma^{-1}) \\ &= u_1^L X S(R_j^{-1}) S(u_a^R) S(R_y) R_\beta u_b^R R_\delta^{-1} \tilde{X} S(R_\gamma^{-1}) S(R_i^{-1}) S(u_2^L) S(R_x) S(R_\alpha) \\ &= u_1^L X S(R_j^{-1}) S(u_a^R) R_y R_\beta^{-1} u_b^R R_\delta^{-1} \tilde{X} S(R_\gamma^{-1}) S(R_i^{-1}) S(u_2^L) R_x R_\alpha^{-1} \\ &= u_1^L X S(R_j^{-1}) S(u_a^R) u_b^R R_\delta^{-1} \tilde{X} S(R_\gamma^{-1}) S(R_i^{-1}) S(u_2^L) \\ &= \epsilon(u^R)\, u_1^L X S(R_j^{-1}) R_\delta^{-1} \tilde{X} S(R_\gamma^{-1}) S(R_i^{-1}) S(u_2^L) = \epsilon(u^R)\, u_1^L X R_j R_\delta^{-1} \tilde{X} S(R_i R_\gamma^{-1}) S(u_2^L) \\ &= \epsilon(u^R)\, u_1^L X \tilde{X} S(u_2^L) \end{aligned} \tag{C.4}$$

as desired.

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